The present wall-function approach is based on the numerical resolution of simplified 1D Turbulent Boundary Layer Equations (TBLE). For the velocity field, given a null wall velocity and the velocity u_m at a distance δ from the wall, the wall shear stress at the wall is computed by solving the following equation:

$$\begin{gathered}\frac{d}{dy}[(\mu+\mu_t)\frac{du}{dy}]=F_u\\ u(y=0)=0\\ u(y=\delta)=u_m\end{gathered} \tag{1}$$

And then computing:

$$\tau_w=\rho u_\tau^2=\mu\left.\frac{du}{dy}\right|_w \tag{2}$$

In order to close equation (1), a model is required for the turbulent viscosity μ_t and, as done by different authors, the currently adopted approach consists in using a mixing length hypothesis:

$$\mu_t=\mu k\, y^+\left(1-e^{-\frac{y^+}{A^+}}\right)^2 \tag{3}$$

Where k=0.41 is the Von Karman constant and A+=19 is the Van Driest constant. An important thing that should be noticed is that, when equation (3) is used for the turbulent viscosity and F_u is assumed constant, equation (1) has a closed, implicit solution, which makes superfluous the actual numerical solution. This solution can be obtained as follows. The equation is first integrated in the wall normal direction y, leading to:

$$(\mu+\mu_t)\frac{du}{dy}=F_u y+\tau_w \tag{4}$$

Then, dividing by ρu_τ^2 and rearranging, leads to:

$$(1+\frac{\mu_t}{\mu})\frac{du^+}{dy^+}=1+F_u^+ y^+ \tag{5}$$

Where:

$$F_u^+=\left(\frac{\mu F_u}{\rho^2 u_\tau^3}\right) \tag{6}$$

Finally, equation (5) is integrated in y+ to give the final solution:

$$u^+(y^+)=\int_0^{y^+}\frac{1}{1+\frac{\mu_t}{\mu}}dy^+ + F_u^+\int_0^{y^+}\frac{y^+}{1+\frac{\mu_t}{\mu}}dy^+ \tag{7}$$

It is known that the Reichardt function:

$$f^+=\frac{1}{k}\log(1+ky^+)+A\left(1-e^{-\frac{y^+}{B}}-\frac{y^+}{B}e^{-\frac{y^+}{C}}\right) \tag{8}$$

Is already a very good approximation for u+(y+) when Fu = 0. This approximation can be improved if, instead of A=7.8 (as in the original Reichardt function), A=8.078 is used (B=11 and C=3 are instead retained at the moment). In order to include also pressure gradient effects one can note that, if f+ in equation (8) is a good approximation of the first integral of u+ in equation (7), then the second integral of u+ in equation (7) can be computed trough an integration by parts. Indeed:

$$u^+(y^+)=\int_0^{y^+}\frac{1}{1+\frac{\mu_t}{\mu}}dy^+ + F_u^+\int_0^{y^+}\frac{y^+}{1+\frac{\mu_t}{\mu}}dy^+ \approx f^+ + F_u^+\int_0^{y^+} y^+\frac{df^+}{dy^+}dy^+ = f^+(1+F_u^+y^+)-F_u^+\int_0^{y^+}f^+dy^+ \tag{9}$$

with the last integral in the previous equation being:

$$\int_0^{y^+}f^+dy^+=\frac{(1+ky^+)\log(1+ky^+)-ky^+}{k^2}+A\left[y^+\left(1+\frac{C}{B}e^{-\frac{y^+}{C}}\right)+B\left(e^{-\frac{y^+}{B}}-1\right)+\frac{C^2}{B}\left(e^{-\frac{y^+}{C}}-1\right)\right] \tag{10}$$

This provides a closed semi-analytical formulation u+(y+) when Fu is different from 0.

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